Geometric phases and the magnetization process in quantum antiferromagnets

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(Received 18 November 2008; published 17 February 2009)

The physics underlying the magnetization process of quantum antiferromagnets is revisited from the viewpoint of geometric phases. A continuum variant of the Lieb-Schultz-Mattis-type approach to the problem is put forth, where the commensurability condition of Oshikawa *et al.* derives from a Berry connection formulation of the system's crystal momentum. We then go on to formulate an effective field theory which can deal with higher dimensional cases as well. We find that a topological term, whose principle function is to assign Berry phase factors to space-time vortex objects, ultimately controls the magnetic behavior of the system. We further show how our effective action maps into a $\mathbf{Z}_2$ gauge theory under certain conditions, which in turn allows for the occurrence of a fractionalized phase with topological order.

DOI: 10.1103/PhysRevB.79.064412 PACS number(s): 75.10.Jm

I. INTRODUCTION

Antiferromagnets subjected to an external magnetic field have attracted considerable attention over the years, largely owing to interesting features which they share with a rather broad class of quantum many-body systems situated on a lattice. The series of quantum phase transitions encountered as the field strength is varied may be viewed as a spin-analog of the superfluid-Mott insulator transitions observed in boson Hubbard models,[1–3] which model, e.g., Josephson junction arrays[4] and optical lattices.[5] Magnetization plateaus emerging at simple fractions of the saturated magnetization are reminiscent of the quantum Hall effects.[6,7] Recent developments reveal the subject to be interconnected with an even richer variety of issues such as the Luttinger theorem,[8] electric polarization in solids,[9,10] and fractionalization of quantum numbers in dimensions greater than one.[11,12] Such fractional natures are also expected to be present in closely related Josephson junction arrays, where they may manifest themselves in the form of fractional ac Josephson oscillations.[13]

In this paper we revisit this problem from yet another perspective, i.e., that of geometric phases.[14,15] The basis of our arguments rests only on rather general properties of Berry connections, topological terms and boson-vortex duality, much of which are valid in any dimension. We aim to shed light on the above body of interrelated phenomena, with particular emphasis on providing a workable format with which to pursue further exotica in quantum spin liquids, anticipated to emerge along this line of study.

II. OUTLINE OF THIS PAPER

In view of the fairly technical nature of the presentation to follow, we wish to highlight in this section its main contents, placing them in context with previous work.

Section III is devoted to a geometrical reinterpretation of the well-known $d=1$ result of Oshikawa, Yamanaka and Affleck (OYA),[6] which yields a quantum-mechanical constraint on the possible values which the magnetization can assume at plateaus, i.e., in a spin-gapped regime. (Throughout this paper the notation d is reserved to denote the spatial dimensionality.) The OYA work (reviewed in Sec. III A) builds on arguments initiated by Lieb, Schulz and Mattis[16,17] (LSM), wherein the central step consists of comparing the crystal momenta of the ground state and a candidate low-lying excited state, constructed from the ground state by applying a slow twist to the spins. (The resulting quantization rule agrees with that obtained by bosonization methods.[18]) Meanwhile there is a simple geometric formulation of the crystal momentum of a *ferromagnetic* spin chain due to Haldane,[19] which incorporates the language of spin Berry phases. We show that this framework can be adapted to our problem involving antiferromagnets, provided the spin moments are partially polarized due to the magnetic field. Carrying out the LSM procedure in this geometric language then simply amounts to evaluating the difference between Berry phases associated with untwisted and twisted spin configurations. We find that this indeed leads immediately to the celebrated OYA rule.

In Sec. IV we show that a similar geometric structure underlies the magnetization properties of antiferromagnets in *arbitrary* spatial dimensions. Here there are no obvious substitutes for the LSM scheme or the bosonization technique (both of which are by design specific to $d=1$), and the generalization of the OYA result to spatial dimensions larger than unity[20] is known to be a subtle problem. Our strategy is to build instead on the semiclassical picture of the previous subsection and to derive a low-energy effective theory for partially polarized antiferromagnets. With the magnitude of the spin polarization essentially fixed by the magnetic field, the long wavelength physics mainly involves the orientational fluctuation of the residual *planar* staggered moment, lying in the plane perpendicular to the field. Symmetry thus dictates that our effective action should be a variant of the quantum *XY* model. A careful derivation confirms this expectation; we find however that the imaginary-time action for this *XY* model also contains a purely imaginary topological term (proportional to the time derivative of the phase variable) whose coefficient is given by $S-m$, with S as the spin quantum number and m the magnetization per site. This perhaps is not surprising if one recalls that a complex-valued low energy action also arises out of the closely related boson-Hubbard model away from commensurate filling factors.[1,21,22]

The later half of Sec. IV thus explores the consequence of complexifying our action by the addition of such a term to the effective *XY* model. We know from earlier work on the hydrodynamical properties of superfluids and superconductors (where actions of the same form appear) that the topological term crucially influences the quantum dynamics of *phase vortices* by subjecting them to a Magnus-type force.[23–25] At the heart of this phenomenon is an Aharonov-Bohm (AB)-type quantum phase interference which renders each vortex to act like a charged particle immersed in a fictitious magnetic field whose strength is fixed by the topological term. (The Magnus force is then simply understood to be a pseudo-Lorentz force.) Since vortex events [point vortices in $(1+1)d$, vortex loops in $(2+1)d$, and vortex sheets in $(3+1)d$] are the most relevant disordering agents of an *XY* model, it is clearly important to see how such an interference effect manifests itself in the present problem. For this purpose we incorporate simple duality techniques which enable us to visualize the physics directly in the language of vortices. (As later explained, the approaches taken here as well as in Sec. V are intimately related to the recent work of Balents *et al.*[26] who (in a context slightly different from ours) run a detailed projective symmetry group analysis to classify the possible ground states which the vortices of the *XY* model can assume.) We find that the pseudo-AB effect generates Berry phase factors associated with the vortex events, which in turn governs (given a specified valued of $S-m$) whether the condensation of these topological defects is allowed as a result of constructive interference, or alternatively, a destructive interference renders the vortex configurations irrelevant. (The former case will imply spin disordering and hence the formation of a magnetization plateau.) It is easy to recognize that the former can happen for $S-m\in\mathbb{Z}$, the condition under which *all* vortex Berry phases become trivial. Defects of arbitrary vorticity (most importantly singly quantized vortices) will then be able to condense. A more subtle situation arises when the value of $S-m$ is a rational number, in which case vortices must be *multiply quantized* in order to condense. With this physical picture (which links Berry phases to the magnetization behavior) in hand, it is also interesting to study the effect of Dzyaloshinskii-Moriya type perturbations which can modify the Berry phase factors. A brief discussion on this issue is given at the end of this section.

In Sec. V we look into a particularly important class of problems where the quantity $S-m$ is a half-odd integer, i.e., $S-m\in\mathbb{Z}+1/2$. The central observation to make here is that the system can dynamically acquire a $\mathbb{Z}_2$ gauge symmetry. We thus come into contact with the work of Sachdev and Park[27] who, in the course of their search for novel spin liquids in the *absence* of a magnetic field arrive at the same effective theory (for easy-plane antiferromagnets)—a lattice *XY* model coupled to an Ising gauge theory with a $\mathbb{Z}_2$-valued Berry phase term. This theory has a phase diagram which accommodates a fractionalized phase,[27,28] and we discuss how this arises within our magnetization problem. Interestingly, we find that this derivation generalizes to the case where $S-m$ is set at other simple rational numbers; for instance when $S-m\in\mathbb{Z}+1/3$, we are lead to a similar effective theory coupled to a $\mathbb{Z}_3$ gauge field with a $\mathbb{Z}_3$-valued Berry phase term. The results of this section thus illustrate the unexpectedly rich phase structure of square lattice antiferromagnets in a magnetic field, suggesting them to be a promising place to seek exotic spin liquid states.

Appendix A supplements Sec. III while the details of the duality methods used in Sec. IV can be found in Appendix B.

III. GEOMETRICAL INTERPRETATION OF THE LIEB-SCHULTZ-MATTIS ARGUMENT

A. Brief summary of Lieb-Schultz-Mattis

The LSM argument[16,17] gives us an important insight into how low-lying excitations in one-dimensional systems are constructed. Let us consider a translationally invariant $d=1$ quantum system with a unique ground state $|\mathrm{G.S.}\rangle$. The key idea of LSM is that the following *twisted* state

$$\hat{U}_{\mathrm{LSM}}|\mathrm{G.S.}\rangle \equiv \exp\left(i\sum_{j=1}^{N}\frac{2\pi j}{N}\hat{S}_j^z\right)|\mathrm{G.S.}\rangle \tag{1}$$

is made up of low-lying excited states (provided that the crystal momentum satisfies a certain condition which will be fixed shortly), and should hence be a useful reference state for extracting information on the low energy spectrum of the system. To evaluate the momentum shift caused by $\hat{U}_{\mathrm{LSM}}$, it is convenient to consider the phase acquired in the process of executing a one-site translation $\hat{T}$ ($\hat{T}\mathbf{S}_n\hat{T}^{-1}=\mathbf{S}_{n-1}$). Since

$$\hat{T}\hat{U}_{\mathrm{LSM}}\hat{T}^{-1} = \exp\left[-\frac{2\pi i}{N}\sum_{j}(S-S_j^z)\right]\hat{U}_{\mathrm{LSM}}, \tag{2}$$

one can readily see that $\hat{U}_{\mathrm{LSM}}$ shifts the crystal momentum P by

$$\delta P_{\mathrm{LSM}} = -\frac{2\pi}{N}\sum_{j=1}^{N}(S-S_j^z) = -2\pi(S-m) \mod 2\pi, \tag{3}$$

where we have introduced the magnetization (density) $m=\sum_j S_j^z/N$. The above expression implies that if $S-m\notin\mathbb{Z}$ the twisted state $\hat{U}_{\mathrm{LSM}}|\mathrm{G.S.}\rangle$ is orthogonal to the ground state $|\mathrm{G.S.}\rangle$, that is, the twisted state consists of excited states of the *finite-size* system. Moreover, it is not difficult to show that for generic Hamiltonians with short-range interactions the energy of the twisted state decreases like $1/N$ as the system size grows $N\nearrow\infty$. Thus we are lead to an interesting dichotomy; if $S-m\notin\mathbb{Z}$, the system in the thermodynamic limit either has gapless excitations over the unique ground state or has several degenerate ground states corresponding to a spontaneous breaking of (translational) symmetry. (The latter possibility becomes relevant when $S-m=p/q$ (where integers p and q are coprime), as one easily sees by repeating the above twist operation q times.) This, combined with a complimentary commensurability argument based on bosonization, yields the so-called *quantization condition* of magnetization plateaus:[6] $Q_{\text{G.S.}}(S-m)\in\mathbb{Z}$ ($Q_{\text{G.S.}}$: period of the *infinite-size* ground state).

In a strict sense, the above LSM argument alone tells us nothing about the presence/absence of plateaus as low-lying excitations created by the LSM twist do not change the total magnetization $\Sigma_j S_j^z$. Moreover, although a similar conclusion has been reached[20] in higher dimensions as well, it is not totally clear why the same sort of quantization condition exists regardless of dimensionality while the LSM argument is valid *only* in one dimension. In an attempt to clarify such issues, and to set the stage for unearthing further exotic properties, we will now reformulate the problem employing the language of Berry phases. Namely we will adopt a geometric approach to evaluating the crystal momenta of magnetic systems,[19] and show that the momentum of Eq. (3), which was central to the LSM argument re-emerges as *an additional Berry phase* which is generated upon application of the twist. An incarnation of this geometric effect will arise in the form of vortex Berry phases in the low energy field theory described in Sec. IV, which can be constructed for arbitrary d.

B. Berry phase argument

We begin by quickly recalling how a spin Berry phase typically arises. Consider a spin-S object represented by a spin coherent state[29] $|\mathbf{n}(t)\rangle$, where the unit vector $\mathbf{n}(t)$ specifies (in a semiclassical sense) the spin's orientation. When the dynamics of the system is such that $\mathbf{n}(t)$ undergoes an adiabatic rotation, returning to its initial value at the end of the excursion, the wave function accumulates a net phase of $S\oint_c d\mathbf{n}(t)\cdot\mathbf{a}(\mathbf{n}(t))=S\omega[\mathbf{n}(t)]$. Here $\mathbf{a}(\mathbf{n}(t))$ is a Berry connection defined by $\langle\mathbf{n}(t)|\mathbf{n}(t)+\delta\mathbf{n}\rangle=\exp[iS\mathbf{a}(\mathbf{n}(t))\cdot\delta\mathbf{n}]$, C is the loop on the unit sphere mapped out by the trajectory $\{\mathbf{n}(t)\}$, and $\omega[\mathbf{n}(t)]$ the solid angle enclosed by C.

One can conceive of situations where a similar phase accumulation is induced along a *spatial* (as opposed to temporal) extent of a many-spin system by a gradual spatial change of $\mathbf{n}$. Such an example was elaborated by Haldane[19] in his study of the crystal momenta of a ferromagnetic spin chain. Here one is concerned with an instantaneous spin configuration written as a direct product of spin coherent states,

$$|\{\mathbf{n}_j\}\rangle \equiv \bigotimes_{j=1}^{N}|\mathbf{n}_j\rangle. \tag{4}$$

As in the previous subsection, information on the crystal momentum P is gained by inspecting how the generator of translation $\hat{T}=e^{i\hat{P}a}$ (a: lattice constant) affects this configuration. However, we should keep in mind that state (4) is in general not an eigenstate of $\hat{T}$, and we will in this semiclassical approach be evaluating the expectation value $\langle\{\mathbf{n}_j\}|\hat{T}|\{\mathbf{n}_j\}\rangle$ instead of the eigenvalue itself.[19] Since by definition $\langle\{\mathbf{n}_j\}|\hat{T}|\{\mathbf{n}_j\}\rangle=\Pi_{j=1}^{N}\langle\mathbf{n}_j|\mathbf{n}_{j-1}\rangle$, [we assume a periodic boundary condition (PBC), where the $(N+1)$th site is identified with site 1] we find, in analogy to the familiar temporal Berry phase, that in the continuum limit

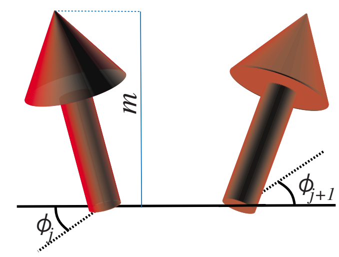

FIG. 1. (Color online) Schematic view of the slowly varying angular variables $\{\phi_j\}$, and the magnetization per site m.

$$\langle\{\mathbf{n}(x)\}|\hat{T}|\{\mathbf{n}(x)\}\rangle = e^{iS\omega[\mathbf{n}(x)]}. \tag{5}$$

The solid angle $\omega[\mathbf{n}(x)]$ is now associated with the loop traced out by the snapshot configuration $\{\mathbf{n}(x)\}$. Equation (5) suggests that the crystal momenta of a $d=1$ ferromagnet is a topological quantity which is best described as a quantal phase defined modulo 2π.

As illustrated in the previous subsection, evaluating the crystal momenta carried by the low-energy states is the central step in a LSM-type scheme.[16] It is therefore tempting to derive the counterpart to Eq. (5) for our present problem, since by introducing such a relation into the LSM program, we can expect to arrive at a geometrical picture underlying the magnetization properties of antiferromagnets. We show below that this is indeed possible. In doing so, however, the following points require clarification. (1) The ground state of a quantum antiferromagnet is generally complicated and cannot be expressed in the form of a spin coherent state as in Eq. (4). However, one can show[30] that if we take the large-S limit, the quantum ground state will generally approach a conventional collinear Néel state. This is the basis of our use of a coherent state ansatz in the following semiclassical treatment. (We will explore in Appendix A what will happen when we relax this ansatz; see the remark of the end of this section.) (2) It is also pertinent to observe that Eq. (5) is sensible only when the spin orientation $\mathbf{n}(x)$ is of a smoothly varying nature. Meanwhile antiferromagnets in an external field would generally have components which vary on the lattice scale. For the Berry phase framework to work, therefore, we must devise a way to represent our system in terms of a coherent state labeled by some slowly varying field.

Let us start then from a canted configuration

$$\mathbf{n}_j=[(-1)^j\cos\phi_j\sin\theta_j,(-1)^j\sin\phi_j\sin\theta_j,\cos\theta_j], \tag{6}$$

where the unit vectors $\{\mathbf{n}_j\}$ represent the orientation of spins partially polarized by a magnetic field applied along the z axis (see Fig. 1). We can identify the unit vector

$$\mathbf{N}_j\equiv(\cos\phi_j\sin\theta_j,\sin\phi_j\sin\theta_j,\cos\theta_j) \tag{7}$$

as a slowly varying (unstaggered) vector field for which the corresponding Berry connection

$$S\mathbf{a}(\mathbf{N}(\mathrm{x}))=\langle\{\mathbf{N}(x)\}|(-i\nabla_{\mathbf{N}(x)})|\{\mathbf{N}(x)\}\rangle \tag{8}$$

is a valid construction. This motivates us to introduce the unitary operator $\hat{U}\equiv\exp[i\Sigma_{j=1}^{N}j\pi\hat{S}_j^z]$ which transforms $|\{\mathbf{n}_j\}\rangle$ into $|\{\mathbf{N}_j\}\rangle$, i.e., $\hat{U}|\{\mathbf{n}_j\}\rangle=|\{\mathbf{N}_j\}\rangle$. Noting that $\hat{T}\hat{U}\hat{T}^{-1}$

$=\exp[i\pi\Sigma_{j=1}^{N}(S-\hat{S}_j^z)]\hat{U}$ (with PBC and N=even assumed), we find that

$$\langle\{\mathbf{n}_j\}|\hat{T}|\{\mathbf{n}_j\}\rangle = \langle\{\mathbf{n}_j\}|\exp\left[i\pi\sum_{j=1}^{N}(S-\hat{S}_j^z)\right]\hat{U}^{-1}\hat{T}\hat{U}|\{\mathbf{n}_j\}\rangle \simeq \exp\left[i\pi\sum_{j=1}^{N}(S-S\cos\theta_j)\right]\langle\{\mathbf{N}_j\}|\hat{T}|\{\mathbf{N}_j\}\rangle, \tag{9}$$

where the last line is a large-S result and is a consequence of the minimal uncertainty property of the spin coherent state.[31] We may now incorporate Haldane's formula, Eq. (5) to obtain

$$\langle\{\mathbf{n}_j\}|\hat{T}|\{\mathbf{n}_j\}\rangle = \exp\left[i\pi\sum_{j=1}^{N}(S-S\cos\theta_j)\right]e^{iS\omega[\mathbf{N}(x)]}. \tag{10}$$

Thus we have separated out the intrinsic part, where the Berry phase appears, from a momentum offset arising from the staggered nature of the ground state. (We pause to observe that in the *absence* of the magnetic field, a similar attempt to extract a Berry phase associated with smoothly varying components would run into difficulties, since that would involve, instead of a simple unitary transformation the simultaneous flipping of all three components of spins residing on every other site, i.e., time reversal operations.) The existence of this magnetization-dependent offset is also easily seen from the exact Bethe-Ansatz solution[32] of the $S=1/2$ XXZ chain.

The next task is to compare Eq. (10) with the corresponding expectation value for a LSM-twisted state. The latter state is defined by $|\{\mathbf{n}_j^{\mathrm{LSM}}\}\rangle \equiv \hat{U}_{\mathrm{LSM}}|\{\mathbf{n}_j\}\rangle$, where $\hat{U}_{\mathrm{LSM}} \equiv e^{i\Sigma_{j=1}^{N}2\pi j/N\hat{S}_j^z}$. We note that $[\hat{U},\hat{U}_{\mathrm{LSM}}]=0$. (Following OYA we are assuming that the interactions are sufficiently local and the Hamiltonian possesses a rotational symmetry around the z axis.) The LSM-twisted counterpart of Eq. (10) is

$$\langle\{\mathbf{n}_j^{\mathrm{LSM}}\}|\hat{T}|\{\mathbf{n}_j^{\mathrm{LSM}}\}\rangle = \exp\left[i\pi\sum_{j=1}^{N}(S-S\cos\theta_j)\right]e^{iS\omega[\mathbf{N}^{\mathrm{LSM}}(x)]}, \tag{11}$$

where $\mathbf{N}^{\mathrm{LSM}}(x)$ differs from $\mathbf{N}(x)$ by a shift of the azimuthal angle, i.e., $\phi_{\mathrm{LSM}}(x)\equiv\phi(x)-\frac{2\pi}{L}x$, $L=Na$. Notice that the twist leaves the offset portion of the momentum unaffected. The right-hand side expressions of Eqs. (10) and (11) will coincide if $S\omega[\mathbf{N}^{\mathrm{LSM}}(x)]=S\omega[\mathbf{N}(x)]+2\pi n$, where $n\in\mathbb{Z}$. Following the usual logic of LSM-type arguments we may interpret this as the semiclassical expression of the necessary condition for the occurrence of a spin gap, i.e., a magnetization plateau. We can make contact with the OYA theory by using the spherical coordinate representation for the solid angle,

$$\begin{aligned} S\omega[\mathbf{N}^{\mathrm{LSM}}(x)] &= S\int_0^L dx[1-\cos\theta(x)]\partial_x\phi_{\mathrm{LSM}}(x) \\ &= S\omega[\mathbf{N}(x)] - \frac{2\pi S}{L}\int_0^L dx[1-\cos\theta(x)] \\ &= S\omega[\mathbf{N}(x)] - 2\pi(S-m), \end{aligned} \tag{12}$$

where $m\equiv\frac{S}{L}\int_0^L dx\cos\theta$ is the magnetization density. This is the Berry-phase derivation of the LSM momentum shift (3). The aforementioned condition thus translates into the quantization rule $S-m\in\mathbb{Z}$. This argument is readily extended to the case where the unit cell consists of $r>1$ sites; the relevant quantity then will be the expectation value of $\hat{T}^r$, leading to the spin-gap condition $rS\omega[\mathbf{N}(x)] = rS\omega[\mathbf{N}^{\mathrm{LSM}}(x)]+2\pi n, n\in\mathbb{Z}$, which in turn yields the OYA quantization rule[6,18] $r(S-m)\in\mathbb{Z}$.

Finally a word on gauge independence is in order, as Eq. (12) involves a particular gauge choice for the monopole vector potential $\mathbf{a}(\mathbf{N}(x))$ (the Dirac string goes through the south pole). One finds that relocating the string to the north pole merely shifts the crystal momentum by $4\pi S$, which is immaterial. Likewise, other gauge choices consistent with the spherical geometry of the target manifold will leave the results unaltered.

As noted above, we have assumed on semiclassical grounds that it suffices to deal with ground states (and their twisted counterparts) which can be expressed as a spin coherent state. One may wonder though whether we can extend the present argument to more generic ground states which are *superpositions* of coherent states. We show that this is indeed possible in Appendix A.

IV. EFFECTIVE ACTION AND DUAL VORTEX THEORY

The findings of the previous section tell us that for $d=1$, a simple semiclassical picture involving partially polarized spin moments is capable of accounting for the results of OYA, once we realize that certain quantum interference effects are at work. The latter are naturally described in terms of Berry phase concepts.

This prompts us to seek a generalization of such geometric interpretations to arbitrary d. Since the LSM approach is now unavailable, we will shift gears and carefully work out an effective low energy theory valid in arbitrary d, invoking once more the semiclassical picture of canted spins. We will find that Berry phase effects are again present, now manifesting themselves in the form of quantum interference among vortex configurations. These interferences can drastically alter the low energy properties of the system. (We will come full circle by later specializing to $d=1$, and identifying a common root that this effect shares with the LSM-type argument presented in Sec. III.) We also discuss possible ways to perturb the system so that the Berry phases (and hence the magnetization properties) are modified.

A. Effective action

We now substantiate the foresaid by deriving a low energy effective action for an antiferromagnet coupled to an external magnetic field. The effective theory, summarized in Eq. (23) below, is valid irrespective of the dimensionality.

For the sake of concreteness, we will consider the following Hamiltonian:

$$\mathcal{H} = J\sum_{\langle i,j\rangle} \mathbf{S}_{\mathbf{r}_i}\cdot\mathbf{S}_{\mathbf{r}_j} + D\sum_i (S^z_{\mathbf{r}_i})^2 - H\sum_i S^z_{\mathbf{r}_i} \qquad (13)$$

on a d-dimensional hypercubic lattice. The first term corresponds to the usual Heisenberg Hamiltonian and the second to the single-ion anisotropy. The external field H is applied in the z direction. If we regard the phase of S^+ and the local magnetization S^z respectively as the Josephson angle ϕ and the boson density n, it is easy to see the similarity to the Josephson junction array:

$$\mathcal{H}_{\mathrm{JJA}} = -\sum_{\langle i,j\rangle} J_{i,j}\cos(\hat{\phi}_{\mathbf{r}_i} - \hat{\phi}_{\mathbf{r}_j}) + \frac{1}{2}\sum_{\langle i,j\rangle} V_{i,j}\hat{n}_{\mathbf{r}_i}\hat{n}_{\mathbf{r}_j} - \sum_i \mu_{\mathbf{r}_i}\hat{n}_{\mathbf{r}_i}, \qquad (14)$$

or, in more general terms, the boson Hubbard (BH) model. In the large-S limit of the model (13), spins assume a canted (or conical) configuration with the XY components aligned in an antiparallel (antiferromagnetic) manner:

$$\mathbf{S}_{\mathbf{r}_j} = S\mathbf{n}(\mathbf{r}_j) = \begin{pmatrix} \sqrt{S^2 - m_j^2}\cos(\mathbf{Q}\cdot\mathbf{r}_j) \\ \sqrt{S^2 - m_j^2}\sin(\mathbf{Q}\cdot\mathbf{r}_j) \\ m_j \end{pmatrix}, \qquad (15)$$

where $\mathbf{Q}=(\pi/a,\ldots,\pi/a)$. When either the field H or the single-ion anisotropy D is finite, a gap opens at $\mathbf{k}=\mathbf{Q}$ and the only gapless mode (a transverse spin wave) appears at $\mathbf{k}=\mathbf{0}$.

Below we extract a low-energy effective theory from the model of Eq. (13), incorporating standard path-integral procedures.[33,34] We illustrate this for the one-dimensional case. (Generalizing to higher dimensions is straightforward.) Guided by the classical solution, we parametrize the fluctuation around the above canted state as

$$n^{\pm}(x) = (-1)^r e^{\pm i\varphi(x)}[\sin\theta^{(0)} + \delta\theta(x)\cos\theta^{(0)}],$$

$$n^z(x) = \cos[\theta^{(0)} + \delta\theta(x)] \approx \cos\theta^{(0)} - \delta\theta(x)\sin\theta^{(0)}, \qquad (16)$$

where $\theta^{(0)}=\cos^{-1}(\frac{H}{2S(2dJ+D)})$. The two equations above each correspond to modifying $e^{i\mathbf{Q}\cdot\mathbf{r}_j}$ in Eq. (15) as $e^{i\mathbf{Q}\cdot\mathbf{r}_j\pm i\varphi(x)}$, and expanding m_j around the average $m=S\cos\theta^{(0)}$. An inspection of the classical Poisson bracket relation for the spin variables suggests that we identify the following as a pair of canonical variables (a denotes the lattice constant):

$$q(x) = \varphi(x), \quad p(x) = -S\sin\theta^{(0)}\delta\theta(x) \equiv a\Pi(x), \qquad (17)$$

which satisfy the equal-time commutator[35]

$$[\varphi(x),\Pi(x')] = i\delta(x-x'). \qquad (18)$$

From the expression

$$S^z_r \approx S\cos\theta^{(0)} + a\Pi(x) + \cdots = m + a\Pi(x) + \cdots, \qquad (19)$$

it is obvious that Π describes the longitudinal fluctuations around the average magnetization m.

Casting these into path-integral form and retaining terms up to second order in φ and Π we obtain the action $\mathcal{S}_{\mathrm{cl}}+\mathcal{S}_{\mathrm{BP}}$ where:

$$\mathcal{S}_{\mathrm{cl}} = \int d\tau\int dx a(2J+D)\Pi^2(r) - \frac{1}{2}S^2\left(1 - \frac{H^2}{4S^2(D+2J)}\right)a\int d\tau\int dx(\partial_x\varphi)^2\mathcal{S}_{\mathrm{BP}} = iS(1-\cos\theta^{(0)})\sum_r\int d\tau\partial_\tau\varphi - i\int d\tau\int dx\partial_\tau\varphi\Pi = i\frac{S-m}{a}\int dxd\tau\partial_\tau\varphi - i\int d\tau\int dx\partial_\tau\varphi\Pi. \qquad (20)$$

The two contributions $\mathcal{S}_{\mathrm{cl}}$ and $\mathcal{S}_{\mathrm{BP}}$ each come from the classical Hamiltonian and the sum over the Berry phases of each spin. Although the first term of $\mathcal{S}_{\mathrm{BP}}$ is a total derivative, it cannot be dropped from the effective action for a reason which will become clear below. Following this intermediate step, we integrate out the Π field to obtain the effective action for the angular field φ:

$$\mathcal{S}_{\mathrm{eff}} = \int dxd\tau\left\{\frac{1}{2}\frac{\zeta}{v^2}(\partial_\tau\varphi)^2 + \frac{1}{2}\zeta(\partial_x\varphi)^2\right\} + i\frac{S-m}{a}\int dxd\tau\partial_\tau\varphi, \qquad (21)$$

where the spin stiffness ζ and the spin-wave velocity v are given by

$$\zeta = aJS^2\left(1 - \frac{H^2}{4(D+2J)^2S^2}\right), \quad v = Ja\sqrt{\frac{4(D+2J)^2S^2 - H^2}{2J(D+2J)}}. \qquad (22)$$

Additional interactions will merely renormalize the values of ζ and v. Note that the φ field appearing in Eq. (21) is defined on a universal covering space of a circle and the last term counts the winding number of the space-time history. In Sec. V we will show that it is also possible to arrive at this term by carefully summing over the spin Berry phase terms $iS\omega[\mathbf{n}(r_j)]$ associated with each lattice site.

In the above, we have assumed that in the transverse (XY, here) direction, the slowly varying degree of freedom is a staggered component with wave vector $\mathbf{Q}$, i.e., possesses at least a short-range antiferromagnetic order. However, the argument goes exactly in the same manner for the spiral (helical) magnets as well; the external field kills two of the three Goldstone modes and the remaining one is described again by φ. Therefore, our effective action (21) is applicable equally well to nonfrustrated and frustrated cases.

Our derivation of $\mathcal{S}_{\mathrm{eff}}$ (21) for $d=1$ readily generalizes to any spatial dimension d, and leads to the following effective action:

$$\mathcal{S}_{\rm eff}[\varphi(\tau,x)] = \mathcal{S}_{\rm top}[\varphi(\tau,\mathbf{r})] + \mathcal{S}_{\rm XY}[\varphi(\tau,\mathbf{r})],$$

$$\mathcal{S}_{\rm top} \equiv i\int d\tau d^d\mathbf{r}\rho\partial_\tau\varphi,$$

$$\mathcal{S}_{XY} \equiv \int d\tau d^d\mathbf{r}\left[\frac{K_\tau}{2}(\partial_\tau\varphi)^2 + \frac{K_\perp}{2}(\nabla\varphi)^2\right], \tag{23}$$

where $\rho \equiv \frac{S-m}{a^d}$, $K_\tau = \zeta/v^2$ and $K_\perp = \zeta$. Equation (23) bears the form of an XY model in $(d+1)$ dimensions supplemented with a topological term. An identical action was previously employed to describe the hydrodynamical properties of superfluids.[23–25] In this analogy, ρ plays the role of a uniform offset value of the superfluid density, as one can read off from Eq. (B3) of Appendix B. There is also an apparent similarity to the low energy theory for the BH model at incommensurate filling factors,[21,22] which is natural in view of the correspondence between Eqs. (13) and (14); we will come back to this connection later. As already mentioned, the topological term $\mathcal{S}_{\rm top}$ is a total derivative and as such will not affect the classical equation of motion. It does however have profound influences on low energy physics when the quantum effects of space-time vortices, to which we now turn, are properly accounted for.

B. Dual vortex theory

The sensitivity of the topological term to vortices is already apparent from the simple observation that a phase winding (in imaginary time) of 2π at spatial position $\mathbf{r}_j$ will yield a nonvanishing contribution $\delta\mathcal{S}_{\rm top} = i2\pi\rho$. To understand the consequence of such effects, it proves convenient to apply to Eq. (23) a standard boson-vortex duality transformation[36] and recast the action in terms of vortex events: pointlike space-time vortices (phase slips) in $(1+1)d$, vortex loops in $(2+1)d$ (world-lines of pointlike vortex-antivortex pairs), and closed vortex surfaces in $(3+1)d$ (world-sheets of vortex rings). For brevity we will generally resort to continuum notations while keeping track of the lattice origin of our model. Following steps well accounted for in the literature,[36,37] we extract the following action (derivations are supplied in Appendix B),

$$\mathcal{S}_{\rm vortex} = \mathcal{S}_{\rm Coulomb} + \mathcal{S}_{\rm BP}^{\rm vortex}, \tag{24}$$

where $\mathcal{S}_{\rm Coulomb}$ represents the $(d+1)$ dimensional Coulombic interaction among space-time vortex objects mediated by the kernel

$$\left(\frac{1}{-\partial^2}\right) \equiv -\left(\frac{1}{2K_\perp}(\partial_\tau)^2 + \frac{1}{2K_\tau}(\nabla)^2\right)^{-1}, \tag{25}$$

and $\mathcal{S}_{\rm BP}^{\rm vortex}$ is the Berry phase associated with vortex events. The latter inherits the information on the "superfluid density" $\rho \propto S-m$ contained in the original topological term and takes the following forms:

$$\mathcal{S}_{\rm BP}^{\rm vortex} = \begin{cases} i\dfrac{2\pi(s-m)}{a}\displaystyle\sum_j q_j a_j^{(0)} & (1+1)d \\ i\dfrac{2\pi(s-m)}{a^2}\displaystyle\sum_{j,\mu} l_{j,\mu}a_{j,\mu}^{(0)} & (2+1)d \\ i\dfrac{2\pi(s-m)}{a^3}\displaystyle\sum_{j,\mu\nu} v_{j,\mu\nu}a_{j,\mu\nu}^{(0)} & (3+1)d, \end{cases} \tag{26a}$$

where $a^{(0)}$ is a solution to

$$\begin{cases} -\epsilon_{\mu\nu}\partial_\nu a_j^{(0)} = \delta_{\mu,\tau} & (1+1)d \\ \epsilon_{\mu\nu\rho}\partial_\nu a_{j,\rho}^{(0)} = \delta_{\mu,\tau} & (2+1)d \\ -\epsilon_{\mu\nu\rho\sigma}\partial_\nu a_{j,\rho\sigma}^{(0)} = \delta_{\mu,\tau} & (3+1)d. \end{cases} \tag{26b}$$

By using the explicit solution to Eq. (26), $\mathcal{S}_{\rm BP}^{\rm vortex}$ can be written compactly as

$$\mathcal{S}_{\rm BP}^{\rm vortex} = \begin{cases} i\dfrac{2\pi(S-m)}{a}\displaystyle\sum_j q_j X_j & (1+1)d \\ i\dfrac{2\pi(S-m)}{a^2}\displaystyle\sum_j q_j A_j^{xy} & (2+1)d \\ i\dfrac{2\pi(S-m)}{a^3}\displaystyle\sum_j q_j V_j^{xyz} & (3+1)d. \end{cases} \tag{26c}$$

The summations on the right-hand side are to be taken over all vortex events, and the q_j's are their vorticities. X_j denotes the spatial coordinate of the jth $(1+1)d$ vortex, A_j^{xy} the area bounded by the projection onto the xy plane of the jth vortex loop C_j in $(2+1)d$. V_j^{xyz} is the volume of the jth vortex surface in $(3+1)d$ projected onto the xyz space (i.e., the net real-space volume occupied by a vortex-ring through its lifetime).

The implications of the vortex Berry phase factors are clear. In $(1+1)d$, the profile of vortex events, when projected onto the spatial coordinate axis consists of points residing on dual sites. The spatial coordinate difference of any pair of vortex events (not necessarily occurring at equal times), $X_i - X_j$, are therefore integer multiples of a. Likewise, in $(2+1)d$ and $(3+1)d$, $A_j^{xy} = a^2 \times$ integer and $V_j^{xyz} = a^3 \times$ integer, respectively. Hence when $S-m \notin \mathbb{Z}$, vortices will create contributions to the partition function that are weighted by oscillatory phase factors, generally leading to a destructive interference (unless events with vorticities of equal magnitudes and opposite signs are confined in pairs). (For the moment we are leaving aside the commensurability effects which set in when $S-m$ is a rational noninteger number.) In other words vortices are, under this condition unable to contribute to the partition function. In $(1+1)d$, for instance, a Kosterlitz-Thouless transition into the plasma phase is prohibited quantum mechanically regardless of the value of the coupling constants.[33,38] When $S-m \in \mathbb{Z}$, in contrast, the phase factors trivialize and the partition function reduces to that of the $(d+1)$ dimensional XY model. Vortices are then able to proliferate and condense, if energetically favorable, and drive the system into a disordered state.

With the hindsight of the results just mentioned, we briefly reflect upon the LSM approach of the previous section and attach to it a simple physical picture. Given that the system in question can basically be regarded as a $(1+1)d$ *XY* model subject to a periodic spatial boundary condition, we shall consider a one-dimensional superfluid (with superfluid density ρ_s) in a ring geometry, a setup often employed to discuss the stability of persistent supercurrents. The LSM twist in this language is nothing but *a phase slip process* which causes the total phase difference along the ring to change by $\pm 2\pi$. It is well known[33,39] that such an event will generate a Galilei boost, i.e., an increment of the center-of-mass momentum of the superfluid by the amount $\Delta P \equiv \int dx \rho_s \partial_x \varphi = \pm 2\pi\rho_s$. The latter leads to a full agreement with the findings of Sec. III provided that we are allowed to equate ρ_s with the coefficient of the topological term, $\rho = \frac{S-m}{a}$. As mentioned earlier, a short inspection of Eq. (B3) in Appendix B confirms that the topological term indeed functions as a source term for the superfluid density, imposing precisely this value. In this way we find that the vortex Berry phases displayed in Eq. (26) and the momentum increment due to the LSM twist share a common origin;[40] the topological information encoded in the term $\mathcal{S}_{\rm top}$ of Eq. (23). Within the present physical picture, the privileged nature of the condition $S-m \in \mathbb{Z}$ may be understood by observing that it allows for *the extra phase winding due to the phase slip to occur without the need for the superfluid to transfer excess momenta to phonons*. (If not for this condition, such transfer will be generically forbidden at zero temperature.) In other words, for this special case it is possible (if energetically favored) for the ground state to be a superposition of different circulation numbers $\frac{1}{2\pi}\oint dx \partial_x \varphi \in \mathbb{Z}$, i.e., a condensate of phase slips.

While we have so far treated the vortices at a "first-quantized" level, the Berry phase factors of Eq. (26) are readily incorporated into second-quantized vortex-field theories,[22,41,42] which provide a framework more amenable to detailed analysis. Referring to the later half of Appendix B for details, we display here their main structures:

$$\mathcal{S}_{\rm dual}^{(1+1)d}[\tilde{\varphi}] = \mathcal{S}_{\rm kin}[\tilde{\varphi}] + y\int d^2r \cos[2\pi(S-m)x + 2\pi\tilde{\varphi}], \tag{27a}$$

$$\mathcal{S}_{\rm dual}^{(2+1)d}[\tilde{b}_\mu,\chi] = \mathcal{S}_{\rm kin}[\tilde{b}_\mu] + \sum_\mu \frac{1}{2\lambda}\int d^3r \times\cos\{2\pi[\Delta_\mu\chi - \tilde{b}_\mu - \delta_{\mu y}(S-m)x]\}, \tag{27b}$$

$$\mathcal{S}_{\rm dual}^{(3+1)d}[\tilde{b}_{\mu\nu},c_\mu] = \mathcal{S}_{\rm kin}[\tilde{b}_{\mu\nu}] + \sum_{\mu<\nu} \frac{1}{2\lambda}\int d^4r \times\cos\{2\pi(\Delta_\mu c_\nu - \Delta_\nu c_\mu) - 2\pi[\tilde{b}_{\mu\nu} + \delta_{\mu y}\delta_{\nu z}(S-m)x]\}. \tag{27c}$$

The kinetic terms denoted by $\mathcal{S}_{\rm kin}$ are quadratic; $\mathcal{S}_{\rm kin}^{(1+1)d}$ is a Gaussian theory, while the higher-dimensional counterparts are Maxwellian terms associated with the dual gauge 1- and 2-forms, $\tilde{b}_\mu$ and $\tilde{b}_{\mu\nu}$. In principle, the action can be supplemented with additional terms consistent with the symmetry requirements of the problem. As explained in Appendix B, $\partial_x\tilde{\varphi}$ $(d=1)$ and the gauge curvatures $\epsilon_{\tau\mu\nu}\partial_\mu\tilde{b}_\nu$ $(d=2)$, $\epsilon_{\tau\mu\nu\lambda}\partial_\mu\tilde{b}_{\nu\lambda}$ $(d=3)$ are directly related to the local magnetization (or in the superfluid analogy, to the superfluid density). The fields χ and c_μ are introduced into the theory to implement the continuity of the vortex current [in $(2+1)d$]/vortex-loop current [in $(3+1)d$].[22,41,42] For each case the Berry phase effect enters as a spatial modulation $2\pi(S-m)x$ within the cosine term, again causing oscillatory behavior unless the aforementioned condition is met. Similar expressions (including higher harmonics terms) have been discussed in $(1+1)d$ using bosonization methods.[2,18]

It is worth noting that for $(2+1)d$, the above expression is identical in form to the free energy of a (classical) $3d$ lattice superconductor, subjected to an external magnetic field of strength $2\pi(S-m)$ (expressed in the Landau gauge). The problem at hand therefore reduces in this case to the energetics of a system of Abrikosov vortices immersed in a dual lattice superconductor; if these vortices manage to destruct superfluidity in this dual theory, it is tantamount to a massless superfluid state of the original *XY* model. If on the other hand superfluidity remains intact in the dual theory, the original theory is disordered, i.e., is massive. This observation enables us to make contact with a related argument due to Lee and Shankar[43] (LS), who also employ a duality technique (which is somewhat different from ours, and in particular do not explicitly emphasize the relevance of Berry phases) to map quantum $2d$ lattice systems into an ensemble of Abrikosov vortices. Following LS, it is natural to expect that when $S-m$ is an integer (or more generally, is a rational number—a case we had not explicitly addressed up to now) the ground state is a periodic vortex lattice commensurate with and pinned by the underlying lattice structure of our spin system. For irrational $S-m$, they will form a floating lattice and destroy the superfluidity of the dual superconductor.

The case where $S-m$ is a noninteger rational number, should however be treated with special caution, since subtle quantum effects may lead to different physics which the naive superconductor energetics can miss.[20] To illustrate what can happen, we go back to the vortex Berry phases of Eq. (26c), and put $S-m=p/q$, $q \ge 2$. Singly quantized vortices/antivortices $(q_j = \pm 1)$ are clearly frustrated; they will suffer the accumulation of an AB-like phase of $\frac{2\pi p}{q}$ upon encircling a dual plaquette. This leads to the destructive interference that we had been describing earlier in this section. Vortices whose winding numbers are integer multiples of q, on the other hand are free of such effects. This leads us to expect that a highly unusual phase can appear, in which only vortices with vorticity $\pm q$ are able to condense; such a tendency can be stabilized by the qth higher harmonics of the cosine term of Eq. (27), which is the leading nonoscillating term in the harmonic expansion. (Needless to say this commensurability effect becomes weak when q is too large.) Since a condensate of higher-winding number vortices can sustain fractional excitations,[44,45] this simple argument points to the possibility of yielding a fractionalized phase in antiferromagnets in an applied magnetic field. Actually an even richer variety of phases are possible in this commensurate case, and can be investigated in its full generality by adopting methods initiated by Lannert *et al.*,[46] and further developed by other authors.[26,27,47,48] (An earlier study of ground state degeneracy and vortex tunneling selection rules in finite-sized lattice systems was undertaken by Altman and Auerbach[13,49]). It is readily seen that there are actually q degenerate low energy modes within the magnetic Brillouin zone for frustrated vortices hopping on a square (dual) lattice under the influence of a $\frac{2\pi p}{q}$-flux piercing each (dual) plaquette. The natural procedure would thus be to construct a Ginzburg-Landau-like action involving q species of vortex fields, and to study the condensation of various composites of such objects. We will not work out the details of this approach here as they can be found in the literature[26] (albeit in different contexts). In Sec. V we will instead focus on the important case where $S-m$ is half-integer-valued and illustrate the nature of possible fractionalized phases which can emerge.

C. Correspondence with the boson Hubbard model

It is instructive to understand how the above conforms with properties of the well-known chemical potential vs tunneling parameter phase diagram of the BH model.[4] The most generic feature of the BH phase diagram are lobes of Mott insulator phases lined up along the chemical potential (μ) axis in the weak tunneling region. The state within each lobe, being incompressible, is characterized by a constant integer-valued boson density N. As evident from the foregoing (see also the discussion of Sachdev[1]), there is a set of correspondences here with the present problem which may be summarized as: $\mu \leftrightarrow H$, $N \leftrightarrow S-m$, Mott insulator $\leftrightarrow$ spin gapped phase, superfluid phase $\leftrightarrow$ gapless phase. The BH counterpart to our problem of determining the gapfulness/gaplessness of the antiferromagnet at fixed $S-m$ thus amounts to examining, at a fixed value of N, whether it is possibility for a superfluid to directly enter a Mott insulator phase upon the approach to the weak tunneling regime. Here we would like to place this analogy on firmer grounds by recalling how Berry phases affect the critical properties of the BH model.

It is known[4] that when $N \in \mathbb{Z}$ the constant-N contour in the superfluid phase merges with the Mott insulator phase (with the same N value) at the tip of the lobe. Meanwhile the contours for noninteger N will disappear into the region in between two adjacent Mott insulator lobes; the fate of these contours (especially those representing commensurate fillings) is a delicate matter[50] and will ultimately depend on the short-range physics. The former situation, in which a superfluid-to-insulator transition is generically possible, is therefore reminiscent of the $S-m \in \mathbb{Z}$ case of the antiferromagnets. The reason behind this similarity can be traced to the behavior of the topological term present in the low energy action; namely, both systems correspond to cases where this term ceases to be effective. We have already discussed in length the irrelevance of Berry phase effects for the $S-m \in \mathbb{Z}$ antiferromagnets. Meanwhile the topological term for

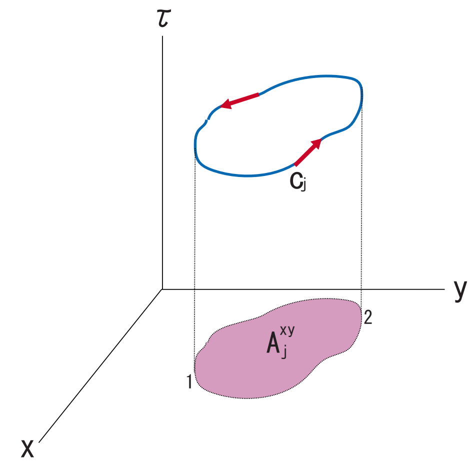

FIG. 2. (Color online) A vortex loop event C_j in $(2+1)d$. Projected onto the xy plane, this corresponds to a vortex-antivortex pair created at point 1, which pair-annihilates at point 2.

the effective action of the BH model is strongly constrained by a U(1) gauge symmetry which the original Hamiltonian possessed.[1,22] This constraint can be used to show that the coefficient of this term vanishes identically right at the lobe tip. Hence the quantum phase transition which takes place along the integer-N contour at this point is in the universality of the $(d+1)$ dimensional XY model, which is the same universality class relevant to the antiferromagnets with $S-m \in \mathbb{Z}$.

D. Effects of antisymmetric interactions

In closing this section we briefly discuss a possible way of modifying the vortex Berry phases we have discussed above. Consider, in the superfluid analogy, allowing a bulk supercurrent $\mathbf{J}$ to flow through a d-dimensional sample; this corresponds to perturbing the spin system with a Dzyaloshinskii-Moriya (DM)-type interaction

$$\mathcal{H}_{\mathrm{DM}} = \sum_{\mathbf{r}} \sum_{a=1}^{d} D_a \hat{z} \cdot (\mathbf{S}_{\mathbf{r}} \times \mathbf{S}_{\mathbf{r}+\hat{e}_a}). \tag{28}$$

It is easy to see that this perturbation induces a nonzero spin current (which translates into a supercurrent), and can be accounted for in the effective theory of Eq. (23) by simply adding the term $iJ_a \nabla_a \varphi$, where $J_a \sim D_a$ is the induced supercurrent. This addition will obviously effect the vortex Berry phases. For instance, while the $(2+1)d$ expression in Eq. (26) basically counts the number of bosons residing within the shaded region of Fig. 2, we must now correct for the migration due to the flow. [Physically, this will cause a change in the quantum dynamics of each vortex. Besides the Magnus force (perpendicular to the velocity of the vortex) coming from the original topological term, vortices will suffer an additional "Lorentz force" which is perpendicular to the supercurrent.] This shifts the Berry phase term by

$$\Delta\mathcal{S}_{\rm BP}^{(2+1)d} = i(2\pi)\sum_{j\in{\rm loops}} q_j(J_x A_j^{y\tau} + J_y A_j^{\tau x}), \tag{29}$$

where $A_j^{y\tau}$ and $A_j^{\tau x}$ are each the area of the projected image of C_j onto the $y\tau$ and τx planes. A similar consideration leads to the following Berry-phase shift in $(1+1)$ dimensions:

$$\Delta\mathcal{S}_{\rm BP}^{(1+1)d} = i(2\pi)J_x\sum_j q_j\tau_j. \tag{30}$$

Now let us consider how the nonzero supercurrent J_a (or, nonzero DM interactions) affects the magnetization process for the simplest $(1+1)$-dimensional case. In accordance with Eq. (27), the low-energy physics in the vicinity of $S-m\in\mathbb{Z}$ may be described by the following sine-Gordon model:

$$\mathcal{S}_{\rm dual}^{(1+1)d}[\tilde{\varphi}] = \mathcal{S}_{\rm kin}[\tilde{\varphi}] + y\int d^2r\,\cos[2\pi(\delta m x + J_x\tau) + 2\pi\tilde{\varphi}] \tag{31}$$

where δm measures a small deviation from $S-m\in\mathbb{Z}$.

By a straightforward extension of the methods used in Refs. 38 and 51, we can carry out a renormalization-group analysis. According to the one-loop renormalization-group equations, there are two different length scales: (i) a length scale $\xi(\delta m, J_x)\equiv 1/\sqrt{(\delta m)^2+(J_x)^2}$ set both by δm and by J_x (or, D) and (ii) another set by the lowest particle (i.e., $\delta S^z\neq 0$) excitation gap $\Delta_{\rm sol}$ in the absence of δm and J_x (if it is finite). When $\Delta_{\rm sol}=0$, nothing competes with δm or J_x and the system is in the gapless (Tomonaga-Luttinger liquid) phase. When $\Delta_{\rm sol}\neq 0$, on the other hand, we have two different possibilities depending on the ratio between $\xi(\delta m, J_x)$ and $\Delta_{\rm sol}^{-1}$; if $\xi(\delta m,J_x)<\Delta_{\rm sol}^{-1}$, renormalization stops well before the system starts feeling the effect of the pinning potential and the system flows into the same gapless phase as above. If $\xi(\delta m,J_x)>\Delta_{\rm sol}^{-1}$, the system is renormalized into another phase. The nature of this phase may be analyzed by mapping the system to a system of (nearly) free fermions;[52] we find that the gap responsible for the plateau is robust against J_x, i.e., the DM interaction D.

V. NATURE OF PLATEAU PHASES–Z_2 GAUGE THEORY

In the previous section, we had seen that for $S-m\in\mathbb{Z}$ vortices do not suffer from interference effects due to the Berry phase (26). As a consequence, the usual $(d+1)$-dimensional XY transition[4,53] separates the gapless XY-ordered (superfluid) phase at large-$K_{x,\tau}$ from the small-K plateau (i.e., insulating) phase where vortices proliferate. For generic incommensurate values of $S-m$, on the other hand, destructive interference among different configurations of topological (vortex) excitations should lead to quite different small-K behaviors. However, it is not easy to capture the fine structures which may emerge as we approach the limit $K\to 0$. For this reason, we utilize a slightly different formulation of the problem[27] more suited to study the strong coupling phases, and attempt to see more closely how the topological excitations destroy the gapless superfluid phases.

The derivation presented below, leading to our effective theories, rely heavily on methods described in Ref. 27. We are led however to interesting differences which we will highlight as they appear. Let us discretize the $(d+1)$-dimensional space-time and put the quantum XY model (23), using the usual Villain form, on the $(d+1)$-dimensional hypercubic lattice:

$$\mathcal{S}_{XY} = \frac{1}{2}K_{XY}\sum_j\sum_{\mu=1}^{d+1}(\Delta_\mu\varphi_j - 2\pi m_{j,\mu})^2. \tag{32}$$

The XY coupling K_{XY} is proportional to $K_{\tau,\perp}$ in Eq. (23). The above XY action should be supplemented by the Berry phase term:

$$\mathcal{S}_{\rm BP} = i(2S)\sum_j \mathcal{A}_{j,\tau}, \tag{33}$$

where $\mathcal{A}_{j,\mu}$ ($\mu=1,\ldots,d+1$; $x_{d+1}=\tau$), the lattice version of the spin gauge field (or equivalently the CP^1 gauge field), is half the area of a spherical triangle enclosed by a triad of unit vectors $\mathbf{n}_0$ (a fixed reference vector which can be chosen arbitrarily), $\mathbf{n}(j)$ and $\mathbf{n}(j+\hat{\mu})$. Unlike the corresponding expression for $S=1/2$ antiferromagnets in the *absence* of magnetic field,[27] $\mathcal{S}_{\rm BP}$ here does not contain an sign-alternating factor, reflecting the canted nature of the classical ground state. In addition to these, we add (by hand) the simplest "Maxwellian term" allowed both by the arbitrariness of $\mathbf{n}_0$ and by the 2π redundancy (compactness) of $\mathcal{A}_{j,\mu}$. The $(1+1)d$ expression (we will be dealing with this case in most of the equations to follow), for example, is

$$\mathcal{S}_{\rm Maxwell}^{(1+1)d} = \frac{1}{2g_m^2}\sum_j(\epsilon_{\mu\nu}\Delta_\mu\mathcal{A}_{j,\mu} - 2\pi q_{j^*})^2. \tag{34}$$

In the above equation, the integer field q_{j^*} resides on the dual lattice site. Carrying out the Poisson resummation and Gaussian integration, we obtain

$$\mathcal{S}_{\rm Maxwell}^{(1+1)d} = \frac{g_m^2}{2}\sum_{j^*}a_{j^*}^2 + i\sum_j(\epsilon_{\mu\nu}\Delta_\mu\mathcal{A}_{j,\nu})a_{j^*}. \tag{35}$$

It is convenient to rewrite the Berry phase term, Eq. (33), as

$$\mathcal{S}_{\rm BP} = i(2S)\sum_j\sum_\mu \delta_{\mu,\tau}\mathcal{A}_{j,\mu} = i(2S)\sum_{j,\mu}(\epsilon_{\mu\nu}\Delta_\mu\mathcal{A}_{j,\nu})a_{j^*}^{(0)}, \tag{36}$$

where we have introduced an off-set gauge field $a_{j^*}^{(0)}$ which satisfies $\epsilon_{\mu\nu}\Delta_\nu a_{j^*}^{(0)}=\delta_{\mu,\tau}$. This can be solved explicitly as, e.g.,

$$a_{j^*}^{(0)} = -j_x^*. \tag{37}$$

As an important physical aside, we note that for the canted spin configuration of Eq. (15), the lattice curl $\varepsilon_{\mu\nu}\Delta_\mu\mathcal{A}_{j,\nu}$ (which usually represents the antiferromagnetic spin chirality fluctuations) is proportional to the vorticity $\epsilon_{\mu\nu}\Delta_\mu m_{j,\nu}$ of the XY spins on a plaquette surrounding j^*. (This is similar to the situation of the easy plane antiferromagnet in the absence of a magnetic field treated in Ref. 27.) In terms of the latter, the Berry phase of Eq. (36) reads

$$\mathcal{S}_{\mathrm{BP}} = i2\pi(S-m)\sum_{j}(\epsilon_{\mu\nu}\Delta_{\mu}m_{j,\nu})a_{j^*}^{(0)}. \tag{38}$$

Notice that if we identify $\epsilon_{\mu\nu}\Delta_{\mu}m_{j,\nu}$ with q_j of the previous section, the above $\mathcal{S}_{\mathrm{BP}}$ exactly coincides with the first line of Eq. (26). This is readily generalized to higher-dimensional cases. For example, in $(2+1)d$ we may write

$$\mathcal{S}_{\mathrm{BP}} = i2\pi(S-m)\sum_{j,\mu}(\epsilon_{\mu\nu\rho}\Delta_{\nu}m_{j,\rho})a_{j^*,\mu}^{(0)} \tag{39}$$

with $(a_{j^*,x}^{(0)},a_{j^*,y}^{(0)},a_{j^*,\tau}^{(0)})=(-j_y^*/2,j_x^*/2,0)$. It is easy to verify that this reproduces the previous result (26c).

Rescaling $a_{j^*}\mapsto(2S)a_{j^*}$ in $\mathcal{S}_{\mathrm{Maxwell}}^{(1+1)d}$ and collecting the terms $\mathcal{S}_{XY}$, $\mathcal{S}_{\mathrm{Maxwell}}^{(1+1)d}$ and $\mathcal{S}_{\mathrm{BP}}$, we arrive at the following action:

$$Z^{(1+1)d} = \sum_{\{m_{j,\mu}\}}\sum_{\{a_{j^*}\}}\int_0^{2\pi}\prod_j d\varphi_j \times\exp\Bigg[-\frac{1}{2}K_{XY}\sum_{j,\mu}(\Delta_\mu\varphi_j-2\pi m_{j,\mu})^2 -\frac{g_m^2}{2}\sum_{j^*}(a_{j^*}-a_{j^*}^{(0)})^2 - i2\pi(S-m)\sum_j(\epsilon_{\mu\nu}\Delta_\mu m_{j,\nu})a_{j^*}\Bigg]. \tag{40}$$

To make further progress we now need to fix the coefficient of the Berry phase term. Here a crucial difference arises with Eq. (53) of Ref. 27, which corresponds to setting $m=0$ (and $S=1/2$) in our Eq. (40). By varying m, we are able to probe through a variety of different effective theories, each characterized by a different set of vortex Berry phase factors. Below we will inspect a few representative cases.

When $S-m\in\mathbb{Z}$, the U(1) gauge field a_{j^*} and the vortices decouple from each other. The a summation in Eq. (40) can then be carried out trivially, to yield the classical XY model in $(d+1)$ $(d=1$ here) dimensions:

$$Z^{(1+1)d} \sim \sum_{\{m_{j,\mu}\}}\int_0^{2\pi}\prod_j d\varphi_j \exp\left[-\frac{K_{XY}}{2}\sum_{j,\mu}(\Delta_\mu\varphi_j-2\pi m_{j,\mu})^2\right] = Z_{XY}^{2d}. \tag{41}$$

Thus in agreement with our arguments of Sec. IV B, we expect to encounter the usual classical XY transition triggered by vortex proliferation as the XY coupling K_{XY} is varied.

A more interesting situation arises for $S-m\in\mathbb{Z}+1/2$. Here the partition function (40) depends on the *parity* of the link variable $m_{j,\mu}$, which leads us to introduce the Ising variable $s_{j,j+\hat{\mu}}$ through[27]

$$m_{j,\mu} = 2f_{j,\mu} + \frac{1-s_{j,j+\hat{\mu}}}{2} \quad (f_{j,\mu}\in\mathbb{Z}, s_{j,j+\hat{\mu}}=\pm 1). \tag{42}$$

Plugging this into Eq. (40) and carrying out the summation over the a_{j^*} explicitly, we obtain[54] the following partition function:

$$Z^{(1+1)d} = \sum_{\{s_{j,j+\hat{\mu}}\}}\int_0^{2\pi}\prod_j d\varphi_j e^{-\mathcal{S}_{\mathrm{Ising}}-\mathcal{S}_{\mathrm{matter}}}, \tag{43}$$

where the action of the quantum $\mathbb{Z}_2$ gauge theory $\mathcal{S}_{\mathrm{Ising}}$ and its coupling to the matter (XY) field each take the form

$$\mathcal{S}_{\mathrm{Ising}} = -K_{\mathrm{IGT}}(g_m)\sum_{\square}\left(\prod_{\square}s_{j,j+\hat{\mu}}\right) + i\pi\sum_j\frac{1-s_{j,j+\hat{\mu}}}{2}, \tag{44a}$$

$$\mathcal{S}_{\mathrm{matter}} = -4K_{XY}\sum_{j,\mu}s_{j,j+\hat{\mu}}\cos\left(\frac{1}{2}\Delta_\mu\varphi_j\right). \tag{44b}$$

The effect of the Berry phase term of Eq. (40) is now encoded in the second term of $\mathcal{S}_{\mathrm{Ising}}$. The gauge coupling K_{IGT}, a monotonically decreasing function of g, is defined through the relation

$$e^{2K_{\mathrm{IGT}}(g)} \equiv \sum_{n\in\mathbb{Z}}e^{-g^2/2n^2}\Bigg/\sum_{n\in\mathbb{Z}}(-1)^n e^{-g^2/2n^2}. \tag{44c}$$

The appearance of the factor 1/2 in the cosine of Eq. (44) is worth noting; it indicates that the Ising gauge field couples to *fractionalized bosons*, whose creation operators can be identified as $b_j^\dagger\sim e^{i/2\varphi_j}$. A theory with the same basic structure, i.e., fractionalized bosons held together by Ising gauge fields, is obtained in (2+1) dimensions as well. Taking the limit $K_{\mathrm{IGT}}=0$, an explicit summation over the gauge variables $s_{j,j+\hat{\mu}}$ leaves us with a theory in which coefficients are doubled wherever $(\Delta_\mu\varphi_j)$ arises; hence we can expect the fractional bosons to be confined in this limit. A more nontrivial phase may emerge by looking into regions with larger K_{IGT} values. It turns out, in fact, that *deconfinement* can occur in (2+1) dimensions for large K_{IGT}, realizing a fractionalized plateau phase with topological order. Such details are most conveniently analyzed by going back to the form of Eq. (40) and performing, in similarity to the previous section, a duality transformation.

Apart from the difference in the expression for $a_{j^*}^{(0)}$, the derivation of the dual theory proceeds according to the prescription of Ref. 27. The final form for $(1+1)d$ reads

$$Z^{1d} = \sum_{\{a_j^*\}}\sum_{\{n_j^*\}}\exp\Bigg(-\frac{1}{2}g_m^2\sum_{j^*}(a_{j^*}-a_{j^*}^{(0)})^2 - \tfrac{1}{2K_{XY}}\sum_{j,j'}\sum_{\mu,\nu}M_{j,j'}^{\mu,\nu}\epsilon_{\mu\lambda}\Delta_\lambda\{n_{j^*}-(S-m)a_{j^*}\} \times\epsilon_{\nu\rho}\Delta_\rho\{n_{j^{*\prime}}-(S-m)a_{j^{*\prime}}\}\Bigg), \tag{45}$$

where $\{n_{j^*}\}$, whose lattice curl is the boson density, is an integer-valued field defined on the dual sites. The kernel $M_{j,j'}^{\mu,\nu}$ governing the density-density interaction reduces to $M_{j,j'}^{\mu,\nu}=\delta_{jj'}\delta_{\mu\nu}$ for the simplest case of spin systems with short-ranged interactions; here it is written in a form which accounts for a more generic Hamiltonian.[55] A similar dual action is also obtained for the $(2+1)d$ case. While we have displayed the form of the dual theory for general $S-m$, we

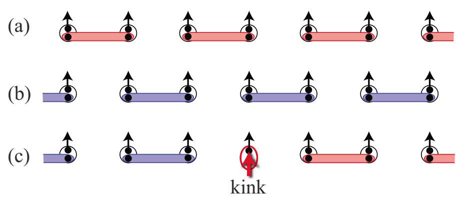

FIG. 3. (Color online) A typical spin state found at an $m=1/2$ plateau in spin-1 chains ($S-m=1/2$). Lattice translation symmetry is broken spontaneously and as a consequence we have two degenerate ground states (a) and (b). Half (arrows) of each spin-1 degree of freedom is frozen by a strong magnetic field and the remaining fluctuating half forms valence-bond solid states (Ref. 57). The lowest excitation is a kink which carries an S^z quantum number of 1/2.

continue to concentrate here on the case $S-m\in\mathbb{Z}+1/2$.

Insight into the phase structure of Z^{1d} (and Z^{2d}) comes from following the analysis carried out in Ref. 27. We begin by considering the limit $g_m\to\infty$. Under this condition, the fluctuations in the a_{j^*} field are quenched $a_{j^*}\mapsto a_{j^*}^{(0)}$ and model (45) reduces (for a diagonal $M_{j,j'}^{\mu,\nu}$) to the dual vortex model of Appendix B [compare with Eqs. (B3) and (B4)]. [Conversely, Eq. (45) extends the standard dual-vortex model treated in Appendix B to allow for fluctuations of the background boson density.] The latter is expected to be in a superfluid (or, XY) phase,[50] which persists into the region $K_{XY}\gg 1$.

Turning next to generic values of g_m, the a_{j^*} field starts to retain its dynamics and induces fluctuations in the background boson density (spin density in the present physical context) $\epsilon_{\tau\lambda}\Delta_\lambda a_{j^*}$. These fluctuations may stabilize a new phase by forming bond-centered orders. [By retracing our steps backward through the sequences described above, one sees that the g_m terms of Eq. (45) and its $(2+1)d$ counterpart are associated with the energies of the spatial links of the direct lattice]. In fact, by recasting our dual theory into a $2d$ height model[27] [a $3d$ frustrated Ising model[56] for the $(2+1)d$ case] or to a field-theoretical model similar to those studied in Ref. 46, it is possible to show that for small K_{XY}, *valence-bond-solid* states are realized in the background of polarized spins moments (m per site), as depicted in Figs. 3 and 4 [for $(1+1)$ and $(2+1)$ dimensions, respectively]. These states support kinklike excitations with the fractional quantum number $\delta S^z=1/2$. The $1d$ version of this valence-bond-solid plateau has been observed numerically.[57] It is interesting to note the apparent similarity of the $\mathbb{Z}_4$-vortex structure shown in Fig. 4 with those studied recently by other authors[58,59] in the context of fractionalized excitations in $(2+1)d$ electron/spin models. In addition to these exotic states, more conventional "collinear" (i.e., CDW) phases may appear when off-diagonal parts of $M_{i,j}^{\mu,\nu}$ are sufficiently large.

Since the $3d$ frustrated Ising model [equivalent to the $(2+1)d$ version of Eq. (45) with $K_{XY}=0$] has an order-disorder transition,[56] the phase diagram in the $(2+1)d$ case is richer than its $(1+1)d$ counterpart; a sufficiently small g_m places the frustrated Ising model in a high-temperature paramagnetic phase (in this language the ordered phase with broken square-lattice symmetry translates into the columnar

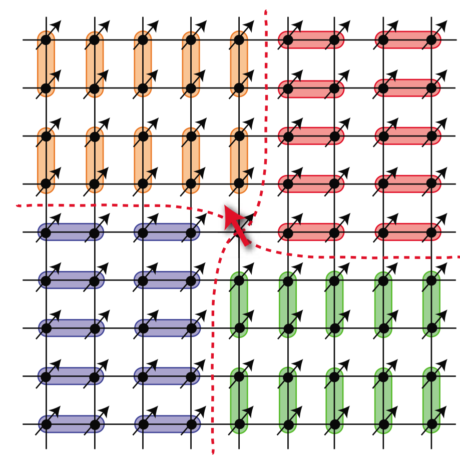

FIG. 4. (Color online) Two-dimensional analog of the spin-1 partially-polarized valence-bond-solid state shown in Fig. 3. Again, the unpolarized "fractions" of the spin-1's form a valence-bond pattern. Dashed lines denote domain walls separating four different valence-bond patterns. At the core of the $\mathbb{Z}_4$-vortex is a fractionalized excitation carrying $S^z=1/2$ (large arrow). The resemblance of the spatial patterns as well as the fractionalized nature of the core excitation with those of Refs. 58 and 59 is evident.

valence-bond-solid phase) which may be identified with the fractionalized phase of the model [Eqs. (44a) and (44b)] with the full space-group symmetry. This suggests that *noncrystalline* plateau states are possible in $d\geq 2$ while in $1d$, as implied by the LSM arguments, the appearance of plateaus is closely tied to the formation of crystalline states. [The dual vortex picture affords us with an alternative simple description of the fractionalized excitations (spinons) which characterize this phase: they are the 2π vortices of the dual Ginzburg-Landau-type field theory which describes the condensate of doubly quantized vortices.[46] The $\mathbb{Z}_4$ kinks in the valence-bond-solid phase, while requiring a subtler analysis,

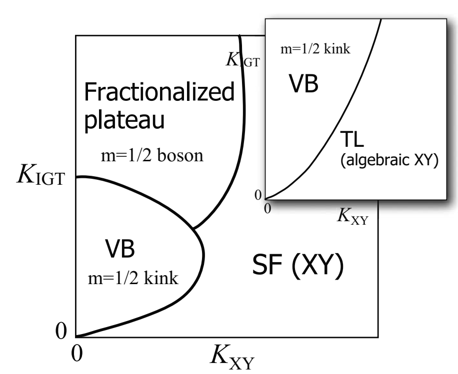

FIG. 5. Schematic phase diagram of the fractionalized boson coupled to $\mathbb{Z}_2$-gauge theory for $(1+1)d$ (inset) and $(2+1)d$. The large-K_{XY} region is dominated by an XY-ordered [or, superfluid (SF)] phase, which, in $(1+1)d$, is replaced with an algebraic Tomonaga-Luttinger (TL) liquid phase. In the small-K_{XY} plateaus region, there is always a valence-bond (VB) phase with broken translation symmetry. Both in $(1+1)d$ and in $(2+1)d$, the VB phase supports topological excitations [kinks in $(1+1)d$ and $\mathbb{Z}_4$ vortices in $(2+1)d$]. On top of this, there is a *featureless* plateau phase with fractionalized (i.e., $\delta S^z=1/2$) bosons in $(2+1)d$.

can also be treated within a similar field theory language.] We summarize the above discussion in the form of a phase diagram, depicted schematically in Fig. 5.

In concluding this section, we note that the two cases considered above, $S-m\in\mathbb{Z}$ and $S-m\in\mathbb{Z}+1/2$, each fall into the class of effective theories for easy plane antiferromagnets (with no magnetic fields and hence $m=0$) with integer and half-integer[27] S. As we have already mentioned, however, the present problem allows us to further extend this approach to other values of $S-m$. For instance the argument applies after appropriate modifications to the case with $S-m=\mathbb{Z}+1/3$; now the original boson is coupled to a $\mathbb{Z}_3$ gauge field, and is fractionalized in such a way that it carries the "charge" $\delta S^z=1/3$. The detailed analysis for general rational values of $S-m$ will be reported elsewhere.

VI. CONCLUSIONS

The message of this paper is twofold: (1) geometric phases provide a natural language to understand how commensurability conditions impose constraints on the generation of magnetization in antiferromagnets, and (2) the analysis based on this language suggests a phase structure which accommodates a wealth of exotic phases.

In the earlier portions of the paper we found that a geometrical interpretation of the LSM approach emerges when we view the spatial modulation of the slow variables as an "adiabatic evolution" along the spin chain. Continuing to focus on geometric phases, we were subsequently led to effective field theories featuring vortex Berry phase factors which are valid in arbitrary dimensions. Taken together, these findings afford us with an explanation of a curious aspect of the LSM-style momentum-counting argument[6,20] which was mentioned back in Sec. III A: its ability to produce the correct criteria for the existence/nonexistence of a magnetization plateau despite the seemingly remote connection between the plateaus and the momenta of charge-neutral ($\delta S^z=0$) excitations. The resolution lies in the fact that the LSM momentum shift and the action $\mathcal{S}_{\mathrm{BP}}^{\mathrm{vortex}}$ governing the low-energy behavior via topological interference stems from a single quantity—the Berry phase of the canted spins.

The approach exploited in the preceding two sections is intended to serve as the basis for seeking as-yet undetected phases whose existence our theory suggests. Indeed the Z_2 gauge theory of Sec. V, which is closely linked to the issue of fractionalization en route the pairing of vortices[44,45] strongly suggests that antiferromagnets in an external magnetic field provides a promising playground to search for fractionalized spin liquids in spatial dimensions higher than one. (A magnetization plateau at $m=1/2$ in translationally invariant $S=1$ Hamiltonians on a $2d$ square lattice, a signature of such a phase, may in this regard be an interesting problem to pursue numerically as well as experimentally.) How the perturbation brought on by spin currents as briefly discussed toward the end of Sec. IV affects the magnetization property is another problem that we believe warrant further investigations. We hope to return to such issues in the future.

ACKNOWLEDGMENTS

We thank Y. Hatsugai, T. Hikihara, J. Inoue, M. Kohno, K. Matsushita, M. Nakamura, Y. Nonomura, M. Oshikawa, and M. Sato for helpful discussions. A.T. acknowledges the hospitality extended by K.-S. Kim at KIAS where an early portion of this study was conducted, and which was later combined with unpublished notes of K.T. He also thanks A. G. Abanov for pointing to several important references on phase slips in $1d$ superconductors. Finally we wish to thank the organizers of the workshop/symposium *Topological Aspects of Solid State Physics* held at ISSP, Kashiwa and YITP, Kyoto where this work was concluded. This work was supported in part by Grants-in-Aid for Scientific Research [Grants No. (C) 18540386, No. (C) 20540386 (A.T.), No. (C) 18540372, and No. (C) 20540375 (K.T.)] from MEXT, Japan. A.T. and X.H. were also supported by the World Premiere International Research Center Initiative of Materials Nanoarchitectonics, MEXT, Japan. K.T. was supported by the global COE program "The next generation of physics, spun from universality and emergence" of Kyoto University.

APPENDIX A: MORE ON THE BERRY PHASE APPROACH TO THE OYA ARGUMENT

In this appendix we discuss how the arguments of Sec. III are to be extended when one wishes to deal with a *superposition* of spin coherent states $|\{\mathbf{n}(x)\}\rangle$. While the case treated in the main text already captures the essential geometric properties inherent in the LSM approach, this generalization intends to fill in the gap between the treatment of the actual ground state of a Heisenberg antiferromagnet (which possesses rotational invariance) with that given in the main text for a spin coherent state which is, in a semiclassical sense polarized in a particular orientation.

Let $|\Psi\rangle$ then be our ground state. We can generally expand this state in the (overcomplete) spin coherent states basis $|\{\phi(x)\}\rangle$, where in accordance with the main text, the low energy physics is described in terms of the angular field $\phi(x)$ specifying the in-plane orientation of the staggered spin component perpendicular to the magnetic field.

$$|\Psi\rangle=\int\mathcal{D}\phi(x)\Psi[\phi(x)]|\{\phi(x)\}\rangle. \tag{A1}$$

The wave functional $\Psi[\phi(x)]=\langle\{\phi(x)\}|\Psi\rangle$ is subject to the normalization condition $\int\mathcal{D}\phi(x)|\Psi[\phi(x)]|^2=1$. We will adopt the action $\mathcal{S}_{eff}[\phi(\tau,x)]$ given in the text [Eq. (23)], though the arguments to follow apply to a wider variety of systems (as long as the coefficient of the topological term remains the same). The Hamiltonian derived from this action is

$$\hat{\mathcal{H}}=\int dx\left\{\frac{1}{2K_\tau}(\hat{\pi}-(S-m))^2+\frac{K_\perp}{2}(\partial_x\phi)^2\right\}, \tag{A2}$$

where $\hat{\pi}(x)=-i\frac{\delta}{\delta\phi(x)}$ is canonically conjugate to $\phi(x)$. The appearance of the gauge fieldlike piece $S-m$ in the kinetic energy term has its origin in the topological term. In the superfluid analogy, it is simply the offset value of the superfluid density. The functional Schrödinger equation reads $\hat{\mathcal{H}}[\hat{\pi}(x),\phi(x)]\Psi[\phi(x)]=E\Psi[\phi(x)]$.

As before we wish to compare the expectation values of the crystal momentum in the ground state and the LSM-twisted state. We recall that for the pure spin coherent state treated in Sec. III, it was for this purpose essential that the quantity $\langle\{\mathbf{n}(x)\}|\hat{T}|\{\mathbf{n}(x)\}\rangle$ picks a Berry phase factor associated with a round trip experienced by the vector $\mathbf{n}$. It turns out however that here the corresponding expression $\langle\Psi|\hat{T}|\Psi\rangle=\int\mathcal{D}\phi(x)\Psi^*[\phi(x-a)]\Psi[\phi(x)]$ is *not* equipped with the anholonomy one expects to find when ϕ is sent on a round excursion. This can be traced to the single-valuedness of $\Psi[\phi(x)]$ which implies that the wave functional be invariant under a 2π shift of $\phi(x)$. Hence as it stands one cannot straightforwardly incorporate the Berry phase effect, which we know from the main text to be crucial in obtaining the correct expression for the crystal momentum.

The resolution comes from seeking an analogy to the quantum mechanical problem of a particle on a ring threaded by an Aharonov-Bohm (AB) flux[60] to which our problem reduces when $K_\perp=0$. As in the treatment of the latter, the AB-like anholonomy can encoded into the wave functional through a (singular) gauge transformation

$$\tilde{\Psi}[\phi(x)]=e^{-i\int dx(S-m)\phi(x)}\Psi[\phi(x)], \tag{A3}$$

which eliminates the AB-gauge field from the Hamiltonian, i.e.,

$$\tilde{\mathcal{H}}=\int dx\left\{\frac{1}{2K_\tau}\hat{\pi}^2+\frac{K_\perp}{2}(\partial_x\phi)^2\right\}, \tag{A4}$$

at the price of introducing a twisted boundary condition for the transformed wave functional [such that a phase change of $-2\pi(S-m)$ is experienced as $\phi(x)$ changes by 2π].

We thus deduce the appropriate generalization of $\langle\{\mathbf{n}(x)\}|\hat{T}|\{\mathbf{n}(x)\}\rangle$ to be

$$\begin{aligned}\langle\tilde{\Psi}|\hat{T}|\tilde{\Psi}\rangle&=\int\mathcal{D}\phi(x)\tilde{\Psi}^*[\phi(x-a)]\tilde{\Psi}[\phi(x)]\\&=\int\mathcal{D}\phi(x)e^{-i\int dx(S-m)\partial_x\phi}\Psi^*[\phi(x-a)]\Psi[\phi(x)],\end{aligned} \tag{A5}$$

where we resorted to continuum notations in the final expression. Notice that the phase factor in the second line reproduces the correct Berry phase while the part $(\Psi[\phi(x+a)])^*\Psi[\phi(x)]$ is single-valued.

Introducing the unitary operator $\hat{U}_{\mathrm{LSM}}=\exp[-i\frac{2\pi}{L}\int dx x\hat{\pi}(x)]$ and using the relation $\hat{U}^\dagger_{\mathrm{LSM}}\hat{T}\hat{U}_{\mathrm{LSM}}=\exp[-i\frac{2\pi}{L}\int dx\hat{\pi}(x)]\hat{T}$, it is easy to see that the counterpart of Eq. (A5) for the LSM-twisted state reads

$$\begin{aligned}\langle\tilde{\Psi}|\hat{U}^\dagger_{\mathrm{LSM}}\hat{T}\hat{U}_{\mathrm{LSM}}|\tilde{\Psi}\rangle&\\&=\int\mathcal{D}\phi(x)\tilde{\Psi}^*\left[\phi(x-a)-\frac{2\pi}{L}a\right]\tilde{\Psi}[\phi(x)]\\&=\int\mathcal{D}\phi(x)\Psi^*\left[\phi(x-a)-\frac{2\pi}{L}a\right]\Psi[\phi(x)]\\&\quad\times e^{i2\pi(S-m)}e^{-i\int dx(S-m)\partial_x\phi}.\end{aligned} \tag{A6}$$

Due to its single-valued nature, $\Psi[\phi(x)]$ basically factorizes into factors of the form $\Sigma_{n(x)\in\mathbb{Z}}C_{n(x)}e^{in(x)\phi(x)}$ coming from each point x. The effect of shifting the angular field $\phi(x)$ by $\frac{2\pi}{L}a$ at each of the L/a sites thus merely amounts to a net factor of unity, i.e., $\Psi[\phi(x)+\frac{2\pi}{L}a]=\Psi[\phi(x)]$. Hence we have

$$\begin{aligned}\langle\tilde{\Psi}|\hat{U}^\dagger_{\mathrm{LSM}}\hat{T}\hat{U}_{\mathrm{LSM}}|\tilde{\Psi}\rangle&=\int\mathcal{D}\phi(x)\Psi^*[\phi(x-a)]\Psi[\phi(x)]\\&\quad\times e^{i2\pi(S-m)}e^{-i\int dx(S-m)\partial_x\phi}.\end{aligned} \tag{A7}$$

A comparison of Eqs. (A5) and (A7) yields the desired quantization rule.

APPENDIX B: BOSON-VORTEX DUALITY IN THE PRESENCE OF A TOPOLOGICAL TERM

Here we provide the main steps leadings to the Berry phases of Eq. (26c), which makes extensive use of the boson-vortex duality transformation (performed at the first quantization level). Though the technique is standard, we feel that it may be worthwhile to illustrate how the presence of a topological term brings about modifications and gives rise to vortex Berry phases which ultimately dominate the physics at low energy. We also sketch how to retain this Berry phase effect when one switches from the first to the second quantized description of the vortices.

1. First quantization

For simplicity we work in the continuum formulation while being careful to take proper account of the Berry phases; carrying out a lattice duality transformation will of course produce the same results. We begin by introducing a Hubbard-Stratonovich auxiliary vector field $J_\mu=(J_\tau,\mathbf{J})$ and rewrite the imaginary-time Lagrangian density of the effective theory $\mathcal{S}[\phi(\tau,\mathbf{r})]=\int d\tau d^d\mathbf{r}\mathcal{L}$ as

$$\mathcal{L}=i\left(J_\tau+\frac{S-m}{a^d}\right)\partial_\tau\phi+\frac{J_\tau^2}{2K_\tau}+i\mathbf{J}\cdot\nabla\phi+\frac{\mathbf{J}^2}{2K_\perp}. \tag{B1}$$

This is followed by a decomposition of the phase field ϕ into components with and without vorticity, $\phi=\phi_{\mathrm{v}}+\phi_r$, where $(\partial_\mu\partial_\nu-\partial_\mu\partial_\nu)\phi_{\mathrm{v}}\neq 0$. The vorticity-free component ϕ_r can be safely integrated out, which yields the constraint

$$\partial_\mu\tilde{J}_\mu=0, \tag{B2}$$

where $\tilde{J}_\mu=J_\mu+\delta_{\mu\tau}\frac{S-m}{a^d}$, and the Lagrangian density becomes

$$\mathcal{L}=i\tilde{J}_\mu\partial_\mu\phi_{\mathrm{v}}+\frac{\left(\tilde{J}_\tau-\frac{S-m}{a^d}\right)^2}{2K_\tau}+\frac{\tilde{\mathbf{J}}^2}{2K_\perp}. \tag{B3}$$

The divergence-free condition, Eq. (B2), is solved explicitly in terms of an auxiliary field whose form depends on the dimensionality:

$$\tilde{J}_\mu = \begin{cases} \epsilon_{\mu\nu}\partial_\nu\varphi & (d=1) \\ \epsilon_{\mu\nu\lambda}\partial_\nu b_\lambda & (d=2) \\ \epsilon_{\mu\nu\lambda\rho}\partial_\nu b_{\lambda\rho} & (d=3) \end{cases} \quad \text{(B4)}$$

where φ, b_μ, and $b_{\mu\nu}(=-b_{\nu\mu})$ are all vorticity-free. Here on the procedures will be described separately for each dimensionality d.

(i) $d=1$: We define the density of space-time vortices ρ^{v} via $(\partial_\tau\partial_x-\partial_x\partial_\tau)\phi_{\mathrm{v}}=\Sigma_j 2\pi q_j^{\mathrm{v}}\delta(\tau-\tau_j^{\mathrm{v}})\delta(x-x_j^{\mathrm{v}})\equiv 2\pi\rho^{\mathrm{v}}$, where $(\tau_j^{\mathrm{v}},x_j^{\mathrm{v}})$ is the space-time coordinate of the jth vortex event [the notation X_j which appeared in Eq. (26) in the main text corresponds here to x_j^{v}.]. After an integration by parts and a shifting of the φ field, $\varphi=\tilde{\varphi}+\frac{S-m}{a}x$, we arrive at

$$\mathcal{L} = \frac{1}{2K_\perp}(\partial_\tau\tilde{\varphi})^2 + \frac{1}{2K_\tau}(\partial_x\tilde{\varphi})^2 + i2\pi\rho_{\mathrm{v}}\left(\tilde{\varphi} + \frac{S-m}{a}x\right). \quad \text{(B5)}$$

Integrating over $\tilde{\varphi}$ we obtain

$$\mathcal{L} = \pi^2\rho^{\mathrm{v}}\frac{1}{-\partial^2}\rho^{\mathrm{v}} + i2\pi\rho^{\mathrm{v}}\frac{S-m}{a}x. \quad \text{(B6)}$$

The first term on the right-hand side is the intervortex Coulombic interaction for which the kernel $1/(-\partial^2)$ was defined in the main text. The second term represents the vortex Berry phase terms, i.e., $i2\pi\int d\tau dx\rho^{\mathrm{v}}\frac{S-m}{a}x=i2\pi\Sigma_i q_i^{\mathrm{v}}\frac{S-m}{a}x_i^{\mathrm{v}}$.

(ii) $d=2$: We start by establishing some notations. The three-vector b_μ in Eq. (B4) is often referred to as the vortex gauge field or the dual gauge field (note the invariance of Eq. (B4) [in $(2+1)d$) with respect to the gauge transformation $b_\mu\to b_\mu+\partial_\mu\chi$]. Dual electric/magnetic fields can be constructed from these gauge fields:

$$\mathbf{E}^{\mathrm{dual}} = (E_x^{\mathrm{dual}},E_y^{\mathrm{dual}}) = (\partial_\tau b_x - \partial_x b_\tau, \partial_\tau b_y - \partial_y b_\tau),$$

$$B^{\mathrm{dual}} = \partial_x b_y - \partial_y b_x. \quad \text{(B7)}$$

They are related to the auxiliary field $\tilde{J}_\mu$ through

$$\tilde{J}_\tau = B^{\mathrm{dual}},$$

$$\tilde{J}_x = -E_y^{\mathrm{dual}}, \quad \tilde{J}_y = E_x^{\mathrm{dual}}. \quad \text{(B8)}$$

Upon introducing the vortex three-current $J_\mu^{\mathrm{v}} \equiv \frac{1}{2\pi}\epsilon_{\mu\nu\lambda}\partial_\nu\partial_\lambda\phi_{\mathrm{v}}$ which couples to the dual gauge field, the Lagrangian density can be rewritten as

$$\mathcal{L} = \frac{1}{2K_\tau}\left(B^{\mathrm{dual}} - \frac{S-m}{a^2}\right)^2 + \frac{1}{2K_\perp}(\mathbf{E}^{\mathrm{dual}})^2 + i2\pi b_\mu J_\mu^{\mathrm{v}}. \quad \text{(B9)}$$

We then divide b_μ (choosing a suitable gauge) into two parts, viz. $b_\mu=\bar{b}_\mu+\delta b_\mu$, where $\bar{b}_\mu$ generates a background magnetic field $\partial_x\bar{b}_y-\partial_y\bar{b}_x=\frac{S-m}{a^2}$ while δb_μ represents the deviation from this offset value, $\delta B^{\mathrm{dual}}\equiv B^{\mathrm{dual}}-\frac{S-m}{a^2}$ $=\partial_x\delta b_y-\partial_y\delta b_x$. The gauge is so chosen that $\bar{b}_\mu$ does not modify the dual electric field $\mathbf{E}^{\mathrm{dual}}$ [a simple choice being $(\bar{b}_\tau,\bar{b}_x,\bar{b}_y)=(0,0,\frac{S-m}{a^2}x)$]. Having set up the necessary notations, we can recast Eq. (B9) as

$$\mathcal{L} = \frac{1}{2K_\tau}(\delta B^{\mathrm{dual}})^2 + \frac{1}{2K_\perp}(\mathbf{E}^{\mathrm{dual}})^2 + i2\pi\delta b_\mu J_\mu^{\mathrm{v}} + i2\pi\bar{b}_\mu J_\mu^{\mathrm{v}}. \quad \text{(B10)}$$

On integrating over δb_μ we are left with

$$\mathcal{L} = \mathcal{L}_{\mathrm{Coulomb}} + i2\pi\bar{b}_\mu J_\mu^{\mathrm{v}}, \quad \text{(B11)}$$

where the first term on the right-hand side is again the Coulombic interaction (now between segments of the space-time vortex loops) which is of the form $\pi^2 J_\mu^{\mathrm{v}}(\frac{1}{-\partial^2})(\delta_{\mu\nu}-\frac{\partial_\mu\partial_\nu}{\partial^2})J_\nu^{\mathrm{v}}$. The second term assigns a Berry phase to each space-time vortex loop, as we will now see. Since the vortex current field J_μ^{v} is nonvanishing only on such loops (with a magnitude equal to the vorticity q_i^{v}), we can convert the space-time integration of this term into a sum of contour integrals along each vortex loop (we use below a short-hand notation $\vec{V}$ to denote a $(d+1)$-vector V_μ):

$$\mathcal{S}_{\mathrm{BP}}^{\mathrm{vortex}} = i2\pi\int d\tau d^2\mathrm{r}\bar{b}_\mu J_\mu^{\mathrm{v}} = i2\pi\sum_j q_j^{\mathrm{v}}\oint_{C_j} ds\frac{dx_{j,\mu}^{\mathrm{v}}(s)}{ds}\bar{b}_\mu(\vec{x}_j^{\mathrm{v}}(s)). \quad \text{(B12)}$$

The parameter $s(\in[0,1])$ is used to specify the distance along each loop contour C_j. Using Stokes' theorem this contour integral can in turn be expressed as an area integral. In vectorial notations $i2\pi\Sigma_j q_j^{\mathrm{v}}\oint_{C_j}d\vec{x}_j^{\mathrm{v}}\cdot\vec{\bar{b}}(\vec{x}_j^{\mathrm{v}})$ $=i2\pi\Sigma_j q_j^{\mathrm{v}}\int_{D_j}d\vec{A}\cdot\mathrm{rot}\,\vec{\bar{b}}$ where $\partial D_j=C_j$ and $d\vec{A}$ is the area element on the two-dimensional domain D_j. Recalling that by definition $\mathrm{rot}\,\vec{\bar{b}}=\frac{S-m}{a^2}\hat{e}_\tau$, the final expression for the Berry phase term is $i2\pi(S-m)\Sigma_j q_j^{\mathrm{v}}A_j^{xy}$, where A_j^{xy} is the area bounded by the xy plane projection of the loop C_j.

(iii) $d=3$: The procedures involved are natural extensions of the $d=2$ case. (Readers seeking more details on the framework specific to $d=3$ can consult the work of Zee.[37]) The Lagrangian density corresponding to Eq. (B9) is

$$\mathcal{L} = \frac{1}{2K_\tau}\left(\epsilon_{ijk}\partial_i b_{jk} - \frac{S-m}{a^3}\right)^2 + \frac{1}{2K_\perp}[(\epsilon_{x\alpha\beta\gamma}\partial_\alpha b_{\beta\gamma})^2 + (\epsilon_{y\alpha\beta\gamma}\partial_\alpha b_{\beta\gamma})^2] + i2\pi b_{\mu\nu}J_{\mu\nu}^{\mathrm{v}}, \quad \text{(B13)}$$

where the suffixes in roman letters stand for spatial indices, and information on the vortex world sheets is encoded in the two-form $J_{\mu\nu}^{\mathrm{v}}\equiv\frac{1}{2\pi}\epsilon_{\mu\nu\lambda\rho}\partial_\lambda\partial_\rho\phi_{\mathrm{v}}$. Within a worldsheet parametrization scheme using a set of two parameters (s,u), the latter can also be written as $J_{\mu\nu}^{\mathrm{v}}$ $=\Sigma_j q_j^{\mathrm{v}}\int ds\int du\,\epsilon^{ab}\partial_a x_{j,\mu}^{\mathrm{v}}\partial_b x_{j,\nu}^{\mathrm{v}}\delta^{(4)}(\vec{x}-\vec{x}_j^{\mathrm{v}}(s,u))$, where a, b are s or u [compare with expression for vortex current employed in Eq. (B12)]. Since each worldsheet is nothing but the space-time trajectory of a vortex loop living in $3d$ space, $J_{\mu\nu}^{\mathrm{v}}$ can be regarded as a vortex loop current. As in the $2+1d$ case we make the decomposition $b_{\mu\nu}=\bar{b}_{\mu\nu}+\delta b_{\mu\nu}$ with the background field satisfying

$$\epsilon_{ijk}\partial_i\bar{b}_{jk} = \frac{S-m}{a^3}, \tag{B14}$$

together with the condition that $\epsilon_{\mu\nu\lambda}\partial_\mu\bar{b}_{\nu\lambda}=0$ when one of the indices is τ. Integrating out $\delta b_{\mu\nu}$, we get

$$\mathcal{L} = \mathcal{L}_{\text{Coulomb}} + i2\pi J^{\text{v}}_{\mu\nu}\bar{b}_{\mu\nu}. \tag{B15}$$

The first term is the current-current interaction between worldsheets, while the second is the Berry phase term, which contributes to the action a sum of surface integrals on each worldsheet. The latter converts, upon an application of the Stokes-Gauss theorem into three-volume integrals of the flux $\epsilon_{\mu\nu\lambda\rho}\partial_\nu\bar{b}_{\lambda\rho}$. Since the only nonzero flux component comes from Eq. (B14), we find that the vortex Berry phase amounts to $i2\pi(S-m)\Sigma_j q^{\text{v}}_j V^{xyz}_j$, where V^{xyz}_j is the three-volume of the interior of the jth worldsheet projected onto the xyz subspace.

2. Second quantization (vortex field theory)

The dual theories derived above can be used to obtain the vortex field theories of Eq. (27). We give a shortcut version of this procedure for the case of $d=2$ which, as mentioned in the main text, is interesting in view of its relation to the work of Lee and Shankar[43] (the steps for the cases $d=1$ and $d=3$ are essentially the same). We first make the following relabeling in Eq. (B10): $\delta b_\mu \to \tilde{b}_\mu$, $\delta B^{\text{dual}} \to \tilde{B}$, and $\mathbf{E}^{\text{dual}} \to \tilde{\mathbf{E}}$. The action is then decomposed into two parts, $\mathcal{S}=\mathcal{S}_{\text{kin}}+\mathcal{S}_{j-b}$, where $\mathcal{S}_{\text{kin}}=\int d^3x[\frac{1}{2K_\tau}\tilde{B}^2+\frac{1}{K_\perp}\tilde{\mathbf{E}}^2]$ is the Maxwellian kinetic energy term, and $\mathcal{S}_{j-b}$ gives the coupling between the vortex current and the dual gauge field. We write the latter as

$$\mathcal{S}_{j-b} = -i2\pi\int d^3x J^{\text{v}}_\mu(\partial_\mu\tilde{\varphi} - \tilde{b}_\mu - \delta_{\mu y}\rho x), \tag{B16}$$

where as before the "superfluid density" in $(2+1)d$ is $\rho=\frac{S-m}{a^2}$, and we have introduced here a scalar field $\tilde{\varphi}$ whose role is to impose the continuity of the vortex current,[22] $\partial_\mu J^{\text{v}}_\mu=0$. We now place our system on a space-time lattice. In so doing, note that the vortex current can be written [as in Eq. (B12)] as $J^{\text{v}}_\mu=\Sigma_j q^{\text{v}}_j \oint_{C_j} ds \frac{dx^{\text{v}}_{j\mu}}{ds}\delta^{(3)}(\vec{x}-\vec{x}^{\text{v}}_j(s))$. In the discretized lattice version of the theory, each segment of the vortex current can be represented by an integer-valued three-vector $t_{j,\mu}=q^{\text{v}}_j\frac{dx^{\text{v}}_{j,\mu}}{ds}$ defined on the dual lattice

The partition function of a grand-canonical ensemble of space-time vortex loops is thus

$$Z = \int \left(\prod_{n,\mu} d\tilde{\varphi}_n d\tilde{b}_{n\mu}\right)\sum_{\{t_{n\mu}\}}\prod_{n,\mu} e^{-S_{kin}[\tilde{b}_{n\mu}]} \times e^{-i2\pi t_{n\mu}(\Delta_\mu\tilde{\varphi}_n-\tilde{b}_{n\mu}-\delta_\mu\rho n_x)}e^{-\lambda t^2_{n\mu}}, \tag{B17}$$

where $n=(n_\tau,n_x,n_y)$ is the site index, and λ the line tension of each vortex loop. Using the Poisson resummation formula, we can trade the sum over $t_{n\mu}$ with an integration over a continuous variable $X_{n\mu}$, at the price of introducing another set of integers $\{m_{n\mu}\}$. After the $X_{n\mu}$ integration is carried out, we obtain

$$Z = \int \left(\prod_{n\mu} d\tilde{\varphi}_{n\mu} d\tilde{b}_{n\mu}\right)\sum_{\{m_{n\mu}\}}\prod_{n\mu} e^{-S_{kin}[\tilde{b}_{n\mu}]} \times e^{-[1/2\lambda 2\pi(\Delta_\mu\tilde{\varphi} - \tilde{b}_{n\mu} - \delta_{\mu y}\rho n_x + m_{n\mu})]^2}, \tag{B18}$$

which is just the Villain form of

$$Z = \int \left(\prod_{n\mu} d\tilde{\varphi}_{n\mu} d\tilde{b}_{n\mu}\right)\prod_{n\mu} e^{-S_{kin}[\tilde{b}_{n\mu}]}e^{-1/2\lambda\cos[2\pi(\Delta_\mu\tilde{\varphi}-\tilde{b}_\mu-\delta_{\mu y}\rho n_x)]}. \tag{B19}$$

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